imperfection geometry of 0.0, 0.5T, T, and 2.0T, respectively. For the three-ply problem, the critical wave numbers were predicted to be (1, 11), (35, 6), (12, 25), and (23, 7) for $h = 2.54$ mm and (2, 3), (7, 2), (5, 3), and (2, 2) for $h = 25.4$ mm. Although the original theory employed to predict these results is based upon Donnell's shell theory, whose limits have been well categorized, one would not expect a major departure from those responses predicted using a three-dimensional theory. Similar results were obtained using the other numerical methods presented in Criterley (1986) and the nonlinear shell programs based on Sander's shell theory using the finite difference (Ball, 1968) and numerical integration (Criterley, 1987) methods. In addition, similar observations were made using the pseudo finite element method program STAGSC-1 (Almroth et al., 1979). Looking only at a w_{max} at a single point may not be easy to identify the buckling mode, especially if there are some 50 to 250 thousand displacements possible and one needs to recognize and identify a buckling pattern that changes with imperfection geometry. The reviewer finds that the reported results have an unusual regularity for the class of problems studied and notes that they differ in behavior from the results obtained from the three aforementioned numerical methods.

Another problem of the paper is that there is an inadequate description of the dynamic analysis methodology. How are the previous displacements of load step and iterations used? Are these quantities considered to be pseudo loads and are accounted for by shifting these to the right-hand side of the system of equations? Are the matrices being inverted and saved or are the system of equations being solved for each time step (step 4)? It is well known that using a Newmark or central difference time operator can produce an unstable solution. Runge-Kutta also falls into this category. Again, is the solution unstable or has the shell reached a bifurcation state? With a nonsymmetric system of equations, the eigenvalue associated with the time step and the matrix equation that represents the equations of motion can be such that a region of instability can be reached—similar to a Mathieu stability phenomenon. The solution could manifest itself with initially a very small error for a considerable length of time. Then, all of a sudden, a large growth in a displacement component is observed. If the solution steps are retraced, the displacement changes are found to exhibit a sign change for each time step. Numerical stability has therefore been encountered and not dynamic instability. One should be cautioned in using only a numerical method using a single observation without some sort of criterion-defining stability, such as applying a variation of energy or a Jacobian determinant. In the reviewer's opinion, this has not been adequately described.

Finally, the title suggest that post-buckling is being investigated. If the brute force method of the numerical solution was used by the authors for all of the classes of analyses presented, no mention has been made as to how the authors circumvent the ill-conditioning of the system of equations just beyond the critical point of bifurcation. Since there are at least two branch paths possible from a bifurcation point (whether static or dynamic), no mention has been made as to how the solution procedure accounts for this possibility.

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Authors' Closure[3]

In the opinion of the authors, investigations of the reviewer are based on a conventional flexural matrix-based finite element analysis stand of view and a previous work by him. Thus, much of his claims are not hold for (does not contribute to) the full three-dimensional theory of elasticity and the proposed solution employed by the authors. He is familiar with a specific solution procedure, and has problems in relating the solution procedure proposed by the authors to what he has in mind.

Due to nonlinearity of the presented governing equations, no explicit stiffness matrix can be introduced unless these equations are linearized. For this reason, instead of using matrix procedures such as the Newton-Raphson method, Broyden-Fletcher-Goldfarb-Shanno method (Bathe, 1996) modal analysis, mode superposition, matrix reduction (Noor and McCom, 1981; Curnier, 1994) or subspace iteration methods, a modified Gauss-Siedel (SOR or successive over-relaxation method that is more rapidly convergent than the GS method) type of solution is used for each iteration within a time step. Thus, a solution vector refreshment is used rather than a matrix procedure. The results are presented for full cylindrical shells with arbitrary axial, radial, and tangential boundary conditions. Due to the variety of the presented examples (examples of static and dynamic buckling under mechanical and thermal loads with symmetric and antisymmetric ply sequences are considered) and due to limitation of the number of the paper pages, additional examples such as a cylindrical shell with asymmetric boundary conditions are not included. Due to the generality of the three-dimensional theory of elasticity, the proposed method is capable of investigating asymmetric and diamond-type buckling. In the presented examples, specifying the load cases was sufficient for understanding the required circumferential and radial boundary conditions. Concepts such as bending-membrane parameters are relevant to the conventional flexural theory that deals with the global behavior of the shell and cannot be defined for the three-dimensional theory of elasticity which is related to the elemental behaviors. On the other hand, it is well known that the prescription of the boundary conditions in the FDM is much simpler than that of FEM. In the paper, FDM is used for replacement of the spatial derivatives. Interlacing or noninterlacing grid points (Brush and Almroth, 1975) can be employed in this procedure and derivative values at the boundary points should be calculated based on the preceding points (Gerald, 1994).

Concepts such as in-plane stresses and rotations about the normals are relevant to global analysis and not to the three-dimensional theory of elasticity. The number of grid points in the axial and circumferential directions in the presented examples are predicted based on the deformed shapes and the number of axial and circumferential wave numbers reported in (Liaw and Yang, 1990; Reddy and Savoia, 1992) and previous works of the authors based on different theories proposed by them (Eslami and Shariyat, 1999; Eslami, Shariyat, and Shakeri, 1998; Shariyat and Eslami, 1999). Generally, grid point distances would be chosen in such a way that no remarkable difference is noticed by decreasing these distances so that effect on the higher modes can be incorporated. Based on this idea, various numbers of grid points are employed where the 150,000 grid points number was a mean value for the different examples. As pointed out earlier, since a vector refreshment scheme rather than a matrix procedure solves the equations, the required computer memory was notably lower. It was mentioned in the paper that the difference in the final results between two numbers of nodes was of the 10^{-12} order. This does not imply that the global error of the numerical procedure itself is of such order.

The authors are aware of interaction of the buckling modes and had confirmed this point in some of their works (Eslami and Shariyat, 1999; Eslami, Shariyat, and Shakeri, 1998; Shariyat and

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Eslami, 1999). The present results are not in contradiction with this point. However, the buckling modes presented by the reviewer are based on the conventional classic methods that have considerable errors in estimating the composite materials (Reddy and Savoia, 1992; Kardomatous and Philobas, 1995). As mentioned in the paper, the generalized concept of dynamic buckling proposed by Budiansky (1974) states that dynamic buckling of a structure occurs when small changes in the magnitude of the loading lead to large change in one of the displacement modes. Location of the buckling point does not alter the global concept of buckling and buckling will occur when the stated conditions are achieved. This criterion is employed widely (for example, Reddy and Savoia (1992) and Gillat et al. (1993)).

In the paper (steps 12 and 13 of the numerical solution scheme) it was emphasized that the results obtained at the end of each time step are added to the displacement or velocity components obtained at the end of the previous time interval (incremental nature of the results also implies this note). As mentioned before, since the governing equations are nonlinear, the idea of establishing any matrix equation is not justified. It seems that the reviewer was expecting a specific numerical procedure and had problems in relating the proposed procedure to what was expected. As may be noted from the proposed solution procedure, an implicit RK (Runge-Kutta) time integration scheme of the solution with iterative modifications is used within each time step. It is well known that the fourth-order RK method has a local error of $(\Delta t)^5$ order and a global error of $(\Delta t)^4$ order (Gerald, 1994). Therefore, for a Δt of 10^{-6} order chosen, the accumulated error in time integration for the whole range of application would be negligibly small. The iteration can actually be of the utmost importance, since any error admitted in the incremental solution at a particular time directly affects, in a path-dependent manner, the solution at any subsequent time. In contrast to explicit methods, implicit methods are unconditionally stable (Curnier, 1994). Time-step size is so chosen that no divergence, no numerical-induced oscillations, and no numerical damping may occur. As can be seen from Figs. 4 and 7 of the paper, the presented results are consistent and monotonous. Therefore, sign change in the results has not occurred. It has been mentioned that buckling is checked by means of the Budiansky criterion. Modal analysis and modal algorithm addressed by the reviewer are limited to smooth solutions of linear systems (Curnier, 1994). Instabilities similar to a Mathieu instability phenomenon that are relevant to influence of the parameters of the governing equations fall into a category of instability analysis known as the Hill-Hsu parametric resonance approach (Simitses, 1990), which is quite different from the proposed time marching approach.

The final paragraphs in the reviewer's note imply that he does not have complete knowledge of the buckling phenomenon and he reviewed the paper from a pure vibrational stand of view. It is well known that bifurcation sounds for perfect structures only (Simitses, 1990). In the presented examples, the studied shells either have initial imperfections or very negligible imperfections are incorporated to check the bifurcation point. Therefore, a buckling path is actually traced and the post-buckling behavior is investigated. Similar studies are done by some well-known references (Liaw and Yang (1990), Reddy and Savoia (1992), and Gillat et al. (1993) are examples).

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Tangential Loading of General Three-Dimensional Contacts[1]

J. Jäger.[2] Recently, Dr. Ciavarella published a paper in the *Journal of Applied Mechanics,* where he presented a generalization of the well-known Cattaneo-Mindlin solution for equal half-spaces with arbitrary contact areas and Poisson's ratio $\nu = 0$. We will shortly explain the background of this generalization, before we discuss some remarks. For vanishing Poisson's ratio, the normal and tangential stress-displacement relations are uncoupled and identical, with exception of a constant factor. Then, each normal solution is also a tangential solution. The two boundary conditions for unidirectional tangential shift are a constant displacement of the stick area and Coulomb's law $q = fp$ in the slip area, where q denotes the tangential traction, p the pressure, and f the coefficient of friction. When the normal pressure p_1 between two bodies in contact on the area C_1 is increased to a new value p_2, the additional pressure $\Delta p = p_2 - p_1$ causes a constant displacement of the old contact area C_1. The increment Δp satisfies the first boundary condition for the tangential problem, when we assume that the old contact area is identical with the stick area and Coulomb's law can be satisfied by setting $q = f\Delta p$. Thus, $q = f\Delta p$ is the solution for the tangential problem. The side conditions of Coulomb's law are also satisfied, as discussed in the article.

In the next paragraphs, we will discuss some points, which can mislead the reader. Some suggestions on future research are presented at the end.

1 The results for axisymmetric contact, summarized by Ciavarella in formulas (29)–(33), have already been published by Jäger (1995a), and presented at the ASME summer meeting (Jäger, 1995b). Table 1 shows a comparison of identical equations by Ciavarella (1998a) and Jäger (1995a). Equation (31) by Ciavarella was written in the form $Q^* = fP(a^*)$ by Jäger (1995a), with $c = a^*$ in Ciavarella's notation. The general solution for plane contact was also published by Jäger (1997a), and some examples have been presented in Jäger (1997b). Dr. Ciavarella requested and received Jäger's papers on April 30, 1998. Nevertheless, in a new publication from July 17, 1998 (Barber and Ciavarella, 1999), Dr. Ciavarella wrote: "A significant generalization of Cattaneo-Mindlin was discovered by Ciavarella . . . for any plane contact problem . . . ". Although this statement was corrected later, Ciavarella did not relate his axisymmetric solutions to Jäger (Jäger, 1995a).

2 "Flat regions are either entirely in full stick or are in full

[1] by M. Ciavarella and published in the Dec. 1998 issue of the ASME Journal of Applied Mechanics, Vol. 65, pp. 998–1003.

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