Then (40), (45), and (54) become

$$\frac{dF_x}{ds} - \rho a_x - cV_x + \frac{d}{ds}\left(\rho_m g z \frac{dx}{ds}\right) = 0 \quad (56a)$$

$$\frac{dF_y}{ds} - \rho a_y - cV_y + \frac{d}{ds}\left(\rho_m g z \frac{dy}{ds}\right) = 0 \quad (56b)$$

$$\frac{dF_z}{ds} + \rho g \simeq 0 \quad (56c)$$

$$\frac{dC_x}{ds} + F_z \frac{dy}{ds} - F_y \simeq 0 \quad (56d)$$

$$\frac{dC_y}{ds} - F_z \frac{dx}{ds} + F_x \simeq 0 \quad (56e)$$

$$\frac{dC_z}{ds} \simeq 0 \quad (56f)$$

$$-C_x + C_z \frac{dx}{ds} = EI \frac{d^2y}{ds^2} \quad (56g)$$

$$C_y - C_z \frac{dy}{ds} = EI \frac{d^2x}{ds^2} \quad (56h)$$

By (56c), (56f), and the boundary conditions at the lower end of the string:

$$F_z|_{s=L} = -\rho_m g L$$
$$C_z|_{s=L} = -T$$

we have

$$F_z = \rho g(L - s) - \rho_m g L \quad (57a)$$

$$C_z = -T \quad (57b)$$

Utilizing (57)

$$s \simeq z$$
$$\frac{d}{ds} \simeq \frac{d}{dz}$$

Then (56a), (56b), (56d), (56e), (56g), and (56h) become

$$\frac{dF_x}{dz} - \rho a_x - cV_x + \frac{d}{dz}\left(\rho_m g z \frac{dx}{dz}\right) = 0 \quad (58a)$$

$$\frac{dF_y}{dz} - \rho a_y - cV_y + \frac{d}{dz}\left(\rho_m g z \frac{dy}{dz}\right) = 0 \quad (58b)$$

$$\frac{dC_x}{dz} + [\rho g(L - z) - \rho_m g L] \frac{dy}{dz} - F_y = 0 \quad (58c)$$

$$\frac{dC_y}{dz} - [\rho g(L - z) - \rho_m g L] \frac{dx}{dz} - F_x = 0 \quad (58d)$$

$$-C_x - T\frac{dx}{dz} = EI \frac{d^2y}{dz^2} \quad (58e)$$

and

$$C_y + T\frac{dy}{dz} = EI \frac{d^2x}{dz^2} \quad (58f)$$

Differentiating (58c) and (58d) once and substituting (58a) and (58b), we obtain

$$\frac{d^2C_x}{dz^2} + \frac{d}{dz}\left\{[\rho g(L - z) - \rho_m g L]\frac{dy}{dz}\right\} - \rho a_y - cV_y + \frac{d}{dz}\left(\rho_m g z \frac{dy}{dz}\right) = 0 \quad (59a)$$

$$\frac{d^2C_y}{dz^2} - \frac{d}{dz}\left\{[\rho g(L - z) - \rho_m g L]\frac{dx}{dz}\right\} + \rho a_x + cV_x - \frac{d}{dz}\left(\rho_m g z \frac{dx}{dz}\right) = 0 \quad (59b)$$

Differentiate (58e) and (58f) twice, substitute (59a) and (59b), respectively, and change the differentiation symbols, and we have

$$EI\frac{\partial^4 y}{\partial z^4} + T\frac{\partial^3 y}{\partial z^3} - \frac{\partial}{\partial z}\left[(\rho - \rho_m)g(L - z)\frac{\partial y}{\partial z}\right] + \rho a_y + cV_y = 0 \quad (60a)$$

and

$$EI\frac{\partial^4 x}{\partial z^4} - T\frac{\partial^3 x}{\partial z^3} - \frac{\partial}{\partial z}\left[(\rho - \rho_m)g(L - z)\frac{\partial x}{\partial z}\right] + \rho a_x + cV_x = 0 \quad (60b)$$

Substituting (1) and (2) into (60), equations (3) are obtained.

**Solution of Function $\tau$**

Let

$$\tau = \exp\left[\lambda\left(\frac{EI}{\rho L^4}\right)^{1/2} t\right] \quad (61)$$

where $\lambda$ is a complex constant. Substituting into (12), we have

$$\lambda^2 + 2\left(\frac{\eta}{2} + i\Omega\right)\lambda + [R_\gamma - \Omega^2 + i(I_\gamma + \eta\Omega)] = 0 \quad (62)$$

where $\eta$ and $\Omega$ are the damping factor and angular velocity parameter as defined by (18) and (19), respectively. The two roots of (62) are

$$\lambda_{1,2} = -\left(\frac{\eta}{2} + i\Omega\right) \mp \left\{\left(\frac{\eta}{2} + i\Omega\right)^2 - [R_\gamma - \Omega^2 + i(I_\gamma + \Omega\eta)]\right\}^{1/2} \quad (63)$$

or

$$\lambda_{1,2} = -\left(\frac{\eta}{2} + i\Omega\right) \mp \left[-\left(R_\gamma - \frac{\eta^2}{4}\right) - iI_\gamma\right]^{1/2} \quad (64)$$

Since

$$\left[-\left(R_\gamma - \frac{\eta^2}{4}\right) - iI_\gamma\right]^{1/2} = \sqrt{\frac{1}{2}}\left\langle -\left\{\left[\left(R_\gamma - \frac{\eta^2}{4}\right)^2 + I_\gamma^2\right]^{1/2} - \left(R_\gamma - \frac{\eta^2}{4}\right)\right\}^{1/2} + i\left\{\left[\left(R_\gamma - \frac{\eta^2}{4}\right)^2 + I_\gamma^2\right]^{1/2} + \left(R_\gamma - \frac{\eta^2}{4}\right)\right\}^{1/2}\right\rangle, \quad \begin{cases} R_\gamma > \frac{\eta^2}{4} \\ I_\gamma \geq 0 \end{cases} \quad (65)$$

(17) follows immediately.

# DISCUSSION

## R. Plunkett[2] and C. H. Wu[3]

Professor Huang and Mr. Dareing are to be congratulated for a very interesting solution to an important technical problem. The careful derivation of the governing equations and boundary conditions is particularly valuable.

The conclusion that the stability is independent of the rotating speed is correct within the assumptions of the problem, but may be a little misleading. If nonlinear terms are included, Kolodner

[2] Professor of Mechanics, Department of Aeronautics and Engineering Mechanics, University of Minnesota, Minneapolis, Minn. Mem. ASME.

[3] Graduate student, Department of Aeronautics and Engineering Mechanics, University of Minnesota, Minneapolis, Minn.



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[9][4] shows that there can be equilibrium displacement shapes corresponding to any rotating speed above first critical, even in the absence of torque. The particular conclusion in this paper is dependent on the linearization of the problem and the absence of unbalance forces. In any actual problem, unbalance will cause deflection at any rotating speed; this deflection will increase greatly at first critical and will continue to increase as the speed is increased, until damping forces a reduction to a lower amplitude, as shown by Caughey [10]. The lack of dependence of shape on rotating speed is controlled by the well-known stabilizing effect of the coriolis term in equation (12), such that the $2i\omega\dot{\tau}$ term exactly balances the $-\Omega^2\tau$ term.

The above discussion is based on solutions for non-stiffened strings. Such solutions are valid for long shafts, i.e., $\alpha = 1000$ in the paper under discussion. The actual solution can be found in terms of that for the string plus exponentially decaying boundary layer solutions at each end [11]. The stabilizing effect of the coriolis coupling leads one to wonder if the constraint of a casing might destabilize a solution. Further work by Wu, which is still in progress and will be reported elsewhere, indicates that the higher modes are indeed unstable and that the only stable position at high speeds is where the pipe is in radial contact with the casing over most of its length.

Because of this and other work, we feel that the singular perturbation methods leading to the exponentially decaying boundary layer are preferable to numerical methods for large $\alpha$. This is borne out by the authors' difficulty with computations at large $\alpha$.

We agree, however, that the numerical solutions may be necessary for medium values of $\alpha$. For small $\alpha$, one should be able to find a perturbation solution with $g$ as a parameter, and the beam with no axial tension as the base solution.

In trying to make a numerical comparison with the value of $R$ given in Table 1, it appears that the asymptotic value of $R/\alpha$ for $\beta = 0$ and large $\alpha$ is about 3. The value usually given is $\left(\frac{2.405}{2}\right)^2 = 1.45$ [12]. Could there be a factor of 2 missing in our interpretation of the dimensionless parameters? This paper is a valuable contribution to the study of the rotating drill string and we await the authors' further work with anticipation.

### Additional References


9 I. I. Kolodner, "Heavy Rotating String—A Non-linear Eigenvalue Problem," *Comm. on Pure and Appl. Math.*, vol. VIII, 1955, pp. 395–408.

10 T. K. Caughey, "Whirling of a Heavy Chain," *Proceedings of the Third U. S. Cong. Appl. Mech., ASME*, 1958, pp. 101–108.

11 C. H. Wu, "Critical Speeds of Rotating Drill Shaft in a Uniform Casing. I—Undamped Case," Technical Report, Dept. of Aero. and Eng. Mech., University of Minnesota, October 21, 1964.

12 L. A. Pipes, *Applied Mathematics for Engineers and Physicists*, McGraw-Hill, Section XVI-7, 1946.


[4] Numbers in brackets designate Additional References at end of this discussion.

## Authors' Closure

We wish to thank Professor Plunkett and Mr. Wu for their discussion. They have stimulated a new analytical problem of drill string mechanics involving a system of nonlinear differential equations. The nonlinear equations can be obtained from the derivation of the equations as given in the Appendix, by retaining the nonlinear terms and by eliminating certain approximations introduced. Besides, the influence of the unbalance in the physical system, of the rotary inertia of a cross section, and of the moving fluid inside the pipe also can be incorporated. This of course constitutes a very challenging problem. Any advance in this study will be a great contribution. Further work on this subject by Wu and others surely will be highly appreciated.

In the absence of torque and damping, the nonlinear solution of Kolodner indicates that a string can rotate at any velocity $\omega > \omega_1$. Thus $\omega$ forms a continuous spectrum as far as the displacement shapes for equilibrium are concerned. According to the linear theory, although $\omega$ has a discrete spectrum in this respect, it forms a continuous spectrum as far as lateral vibration is concerned.

It is true that for large value of $\alpha$, the singular perturbation method offers an alternative to the numerical method presented. However, it is believed that the numerical difficulties in the latter can be circumvented by the use of the delta matrices [13].

From a solution to the second mode, it has been found that to stabilize the second mode requires more damping than the first. This somehow agrees with Wu's finding that the higher modes are really unstable.

The numerical difference between the solution in [12] and the data presented arises from the difference in the lower end boundary conditions. Whereas the solution in [12] refers to a string supported at top but free at bottom, the data presented apply to a pipe supported at top and guided vertically at the bottom. In a similar investigation [14], it has been shown that for the problem in [12] the numerical method yields an asymptotic value of $R/\alpha$ very close to the value given in [12]. This is particularly true for the lower modes.

We also appreciate the oral discussion given by H. B. Woods at the Conference. The variation of torque along the rod he mentioned can be easily incorporated in the solution, if the torque is a polynomial function of $z$. The symbol $\rho$ in the expression of $\Omega$ and $\eta$ would be better interpreted as the mass participating in the lateral motion, which includes the mass of the rod and the mass of the mud inside the pipe, and the virtual mass of the mud outside the pipe.

### Additional References


13 H. Furke, "Bestimmung vo Balkenschwingungen mit Hilfe des Martrizenkalküls," *Ingr.-Arch.*, vol. 23, 1955, pp. 329–384.

14 Tseng Huang and D. W. Dareing, "Frequencies of Lateral Vibration and Critical Weights of Long Vertical Rods," (in preparation).




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