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Discussion

Equilibrium Configurations of Cantilever Beams Subjected to Inclined End Loads[1]

C. Y. Wang[2]. The authors may not be aware of a paper on the same topic published more than ten years ago (Wang, 1981). The parameters of Wang, (K, α) are exactly the same as those of the present paper (α, β). Both papers showed the number of nonunique solutions for given parameters, e.g., Fig. 2 of Wang (1981) and Tables 1–4 of the present paper, the latter covering a wider range.

Aside from the asymptotic analyses, the numerical method presented by Wang (1981) seems to be much more efficient than the traditional method of using elliptic functions.

References

Wang, C. Y., 1981, "Large Deflections of an Inclined Cantilever with an End Load," *International Journal of Nonlinear Mechanics*, Vol. 16, pp. 155–164.

Author's Closure[3]

I would like to thank Professor Wang for providing the additional reference, Wang (1981). This paper was not seen by the authors during their literature search and was not referenced in their manuscript. The problem investigated by the authors was studied in another form in Wang's paper and was solved using a completely different approach. The results of the two studies also varied, but both showed the existence of the multiple equilibrium solutions for a cantilever beam subjected to a static end load.

In the work of Navaee and Elling (1992), a powerful technique was discussed for the detection and determination of all possible equilibrium configurations of a particular beam subjected to any specific end load. To the best knowledge of the authors, this efficient technique has not been previously shown in any publication. It should be mentioned that the method shown by Wang (1981) can also be employed to obtain the results presented by the authors, but this method is not very efficient mainly because of the following reason.

In the work of Wang, the slope at the free end of the beam for a specific beam and loading condition, is not known at the outset. Therefore, the differential equation of the beam should be numerically solved for a series of guessed values of the free end slope until the correct solution, satisfying the boundary conditions is determined. This requires a lot of computational effort. It should be mentioned that in an earlier work of the author (Navaee, 1989), equilibrium configurations of beams were computed using several different procedures, one of which was similar to the numerical method discussed in Wang's paper. It was determined by Navaee (1989) that the method employing the elliptic integrals was by far the most efficient technique used, because it eliminated a lot of unnecessary computational work.

Aside from the efficiency and usefulness of the technique used in the author's paper, in my view, it is also mathematically interesting to recognize how the well-known solutions expressed in terms of elliptic integrals can be employed to detect and determine the multiple equilibrium configurations of a beam.

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Navaee, S., and Elling, R. E., 1992, "Equilibrium Configurations of Cantilever Beams Subjected to Inclined End Loads," ASME Journal of Applied Mechanics, Vol. 59, pp. 572–579.

Wang, C. Y., 1981, "Large Deflections of an Inclined Cantilever with an End Load," *International Journal of Nonlinear Mechanics*, Vol. 16, pp. 155–164.

Two-Dimensional Rigid-Body Collisions With Friction[4]

W. J. Stronge.[5] The authors analyze two-dimensional or planar collisions between two rough rigid bodies using Routh's graphical method. They attempt to calculate the loss in kinetic energy during collision using a coefficient of restitution defined by either Newton's kinematic law of restitution e or Poisson's hypothesis e_0. In this discussion I prove that the consequences of these definitions are impractical if the collision is eccentric and the direction of sliding changes during contact; e.g., these definitions yield coefficients that depend on orientation of the bodies, friction, and the initial speed of sliding in addition to internal sources of dissipation. Furthermore, elastic collisions are represented by $e = e_0 = 1$ only if the direction of sliding does not change!

In this analysis each rigid body has a velocity for the point of contact. Routh's method gives changes in the difference between these velocities during collision as a function of tangential and normal components of impulse, P_x and P_y, respectively. Changes in relative velocity between two coincident contact points are obtained by implicitly assuming that these points are separated by a deformable particle. The analysis divides the collision process into a period of compression in which the bodies are approaching each other and a period of

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[4]By Y. Wang and M. T. Mason and published in the Sept. 1992 issue of the ASME Journal of Applied Mechanics, Vol. 59, pp. 536–642.

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restitution in which they are moving apart. The contact points can have a tangential component of relative velocity called sliding S, so the collision process is also divided into a period of sliding in each direction, $S>0$ and $S<0$. During sliding, changes in the normal and tangential components of impulse are related by Coulomb's law, $dP_x = -\mu\, dP_y \operatorname{sgn}(S)$. The question is, what determines when the collision process terminates or how does one calculate the final velocities and system kinetic energy at separation?

The terminal condition is determined by a coefficient of restitution. At least three definitions have been proposed: Newton's law of restitution e (the kinematic coefficient); Poisson's hypothesis e_0 (the kinetic coefficient); and the energetic coefficient of restitution e_* proposed by Stronge (1990). These three definitions are equivalent unless the bodies are rough, the impact configuration is eccentric, and initial sliding halts or reverses before separation. For impact conditions where these definitions are distinct, Wang and Mason compared results obtained with the kinematic and kinetic coefficients of restitution. For $e=1$ they found a range of values for coefficient of friction μ where their calculation indicated that the system gained kinetic energy during collision; in contrast, their calculations for $e_0=1$ always exhibited energy dissipation. Since impact calculations employing the kinetic coefficient never suffer from anomalous increases in kinetic energy, the authors concluded that the coefficient of restitution should be defined in accord with Poisson's hypothesis.

Although calculations of separation velocity obtained with Poisson's hypothesis do not exhibit increases in kinetic energy for any collision, they suffer from precisely the same defect as calculations that employ Newton's kinematic law of restitution; i.e., the coefficients of restitution e and e_0 are not representative of energy absorbed by nonfrictional sources of dissipation. In those cases where it matters, Newton's law of restitution requires too much work to be done by elastic strain energy during the restitution period while Poisson's hypothesis provided too little work from this source. On the other hand, the energetic coefficient of restitution e_* is defined according to this work so it correctly accounts for the part of the total kinetic energy loss due to friction and the part due to internal irreversible deformation. Of course calculations that show energy gain or loss assume that for elastic impact the respective coefficients of restitution are constants. Here lies the fallacy in Newton's law and Poisson's hypothesis. Since there are no increases in total kinetic energy in any experiment, one can prove that both e and e_0 are functions of the coefficient of friction μ and the process of sliding rather than being constants. While there is nothing inherently wrong with coefficients $e(\mu, S_o, B_i)$ and $e_0(\mu, S_o, B_i)$, they are useless in practice because they depend on too many parameters of the system. Moreover, for *elastic collision* between rough rigid bodies in any eccentric configuration, these coefficients have values $e<1$, $e_*=1$, and $e_0>1$ if slip halts before separation. Thus for collisions where slip halts and then either reverses or sticks, *only the energetic coefficient* e_* *is useful for calculating changes in velocity that occur during impact.*

These assertions can be supported by a brief analysis. The authors defined a normal component C and a tangential component S of relative velocity of the contact point on one body from that on the other. At incidence $C(0)\equiv C_0<0$ and $S(0)\equiv S_0>0$. With Coulomb's law of friction the equations of relative motion of the bodies are

$$\begin{aligned} dC/dP_y &= \mu s B_3 + B_2 \\ dS/dP_y &= -\mu s B_1 - B_3 \end{aligned} \qquad (a)$$

where magnitude of the normal component of impulse P_y is a monotonously increasing independent variable, $s=\operatorname{sgn}(S)$ and B_1, B_2, B_3 depend on the inertia of the system. (Incidentally, the authors expression for B_3, Eq. (21), contains an error in sign; both terms should be positive.) Noting that $B_1>0$ and $B_2>0$, these equations result in the following range of parameter values for each different collision process:

$$\begin{aligned} -\mu^{-1}B_2 < B_3 < \mu^{-1}B_2 &\quad \text{gives } dC/dP_y>0 \\ -\mu B_1 < B_3 < \mu B_1 &\quad \text{slip stick if sliding halts} \\ \mu B_1 < B_3 &\quad \text{reversal if sliding halts.} \end{aligned} \qquad (b)$$

Reversal can occur only if $B_3>0$. For the initial period of sliding, integration of (*a*) gives the authors Eq. (17) and (18). These equations indicate that sliding halts or reverses during compression if

$$-\frac{S_0}{C_0} < \frac{B_3+\mu B_1}{B_2+\mu B_3}, \qquad (c)$$

while sliding halts or reverses during restitution if

$$\frac{B_3+\mu B_1}{B_2+\mu B_3} < -\frac{S_0}{C_0} < \left(\frac{B_3+\mu B_1}{B_2+\mu B_3}\right)\frac{\hat{P}_y}{\overline{P}_y}, \qquad (d)$$

where $\overline{P}_y$ is the normal impulse at transition from compression to restitution and $\hat{P}_y$ is the terminal normal impulse at separation. By setting $S=0$ in (17), the normal impulse when sliding vanishes $\tilde{P}_y$ is given by

$$\tilde{P}_y = S_0(B_3+\mu B_1)^{-1}. \qquad (e)$$

At this impulse the normal component of relative velocity is $C(\tilde{P}_y) = C_0 + (B_2+\mu B_3)\tilde{P}_y$.

Consider an initial sliding speed S_0 such that sliding reverses during compression. Before sliding reverses, the normal impulse does work $\tilde{W}_y$ on the bodies, while during the remaining part of compression after reversal, this impulse does work $\overline{W}_y - \tilde{W}_y$. The work done during each separate period of slip can be calculated from (56) but notice that this equation is applicable only for unidirectional slip (Stronge, 1992).

$$\begin{aligned} 2\tilde{W}_y &= -(B_2+\mu B_3)\tilde{P}_y^2 - 2(B_2-\mu B_3)(\overline{P}_y-\tilde{P}_y)\tilde{P}_y \\ 2\overline{W}_y &= -(B_2-\mu B_3)\overline{P}_y^2 - 2\mu B_3\tilde{P}_y^2 \end{aligned} \qquad (f)$$

The normal impulse that terminates compression $\overline{P}_y$ is given by (46). Following compression there is a period of restitution $\hat{P}_y - \overline{P}_y$. The velocity components at separation and work done by normal impulse during restitution can be expressed as

$$\begin{aligned} \hat{C} &= (B_2-\mu B_3)(\hat{P}_y-\overline{P}_y) \\ \hat{S} &= -(B_3-\mu B_1)(\hat{P}_y-\tilde{P}_y) \\ 2(\hat{W}_y-\overline{W}_y) &= (B_2-\mu B_3)(\hat{P}_y-\overline{P}_y)^2. \end{aligned} \qquad (g)$$

The collision process terminates at normal impulse $\hat{P}_y$ given by a coefficient of restitution. The energetic coefficient of restitution e_* is defined as follows: *The square of coefficient of restitution* e_*^2 is the ratio of elastic strain energy released at the contact point during restitution to the energy absorbed by internal deformation during compression (Stronge, 1991). For negligible tangential compliance this energy ratio can be calculated as the negative of work done by the normal component of contact force (or impulse)

$$e_*^2 = -(\hat{W}_y - \overline{W}_y)/\overline{W}_y \qquad (h)$$

where total work done by the normal component of impulse $\hat{W}_y$ is the negative of the energy absorbed by nonfrictional sources of energy dissipation. If the impact configuration and initial velocities are such that sliding reverses during compression, this gives a terminal impulse $\hat{P}_y$ where

$$\hat{P}_y = \overline{P}_y + e_*\left\{\frac{2\mu B_3\tilde{P}_y^2 + (B_2-\mu B_3)\overline{P}_y^2}{B_2-\mu B_3}\right\}^{1/2}. \qquad (i)$$

Hence, for the energetic coefficient, the total work done by normal impulse is

$$2\hat{W}_y = -(1-e_*^2)[2\mu B_3\tilde{P}_y^2 + (B_2-\mu B_3)\overline{P}_y^2] \qquad (j)$$

whereas for Poisson's hypothesis,

$$2\hat{W}_y = -(1-e_0^2)[2\mu B_3\tilde{P}_y^2 + (B_2-\mu B_3)\overline{P}_y^2] - 2e_0^2\mu B_3\tilde{P}_y^2. \qquad (k)$$

Thus, if e_0 is independent of friction, Poisson's hypothesis exhibits a loss of kinetic energy due to internal irreversible deformation in addition to the energy absorbed by friction. On the other hand, for Newton's kinematic law of restitution, the normal component of impulse does work,

$$2\hat{W}_y = -(1-e^2)[2\mu B_3\tilde{P}_y^2 + (B_2-\mu B_3)\overline{P}_y^2] + 2e^2\mu B_3\tilde{P}_y^2\left[\frac{2\overline{P}_y}{\tilde{P}_y} + \frac{2\mu B_3}{(B_2-\mu B_3)} - 1\right] \qquad (l)$$

Here, if e is independent of friction, there is more energy recovered during restitution than was absorbed during compression. Consequently, with friction and slip reversal, $e = e_0 = 1$ do not represent elastic collisions! A similar analysis shows that this conclusion applies also to eccentric collisions which slip stick.

References

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Stronge, W. J., 1992, "Energy Dissipated in Planar Collision," ASME Journal of Applied Mechanics, Vol. 59, pp. 681–682.

Authors' Closure[6]

Stronge (1990) develops a new definition of restitution, which is recapitulated in his discussion above. Stronge's definition of restitution is based on an underlying model of the material behavior. This construction provides a means of relating the coefficient of restitution of the elasticity of the materials.

The main issue raised in the discussion is whether Poisson's definition of restitution must be rejected. Stronge takes the position that Poisson's definition violates basic principles, and is untenable. We do not find the argument compelling. If we accept Stronge's modeling assumptions, then Poisson's definition does not work. But might there be another model of the underlying material interactions, departing from Stronge's assumptions, that supports Poisson's definition? The existence of a satisfactory model seems unlikely, but at this time the question remains open. The terms "fallacious" and "defective" should be reserved for ideas that are proven to be inconsistent with generally accepted principles.

A second issue is whether Poisson's coefficient of restitution is a material constant. Stronge suggest that it is actually a function of initial conditions, coefficient of friction, etc. We think the resolution of this issue is quite simple. One should choose the model one prefers, and then stick with it. If one chooses Poisson's definition, then e is a material constant, and an elastic collision is defined to be $e = 1$. If instead one chooses Stronge's definition of an elastic collision, it would be senseless to use Poisson's coefficient of restitution.

The primary issue should be which definition to prefer. Since Stronge's definition of restitution is better founded than Poisson's or Newton's, and in the absence of any apparent drawbacks, the obvious conclusion is that everybody should adopt Stronge's model. We would like to observe that most of our paper is virtually unaffected by this choice. In particular, (1) the use of Routh's method is trivially extended to handle Stronge's restitution; (2) the concept of tangential impact is unaffected; (3) the taxonomy of rigid body impact is unaffected; and (4) the comparison of Poisson's definition to Newton's is unaffected, though less interesting.

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The Complementary Potentials of Elasticity, Extremal Properties, and Associated Functionals[7]

R. T. Shield[8] and S. J. Lee.[9] The paper by Wempner (1992) gives several references related to variational principles for finite elastic deformations but omits references to the work of Lee and Shield (1980a, 1980b). In the first paper (1980a) we derive a complementary energy principle which uses trial functions for the actual deformation gradient. This approach avoids difficulties associated with inversion of the constitutive relation which in general involves rotations as well as strains. However, the trial functions used in the complementary energy principle satisfy nonlinear equilibrium equations in general, and this causes varying degrees of difficulty in applying the principle depending on the form of the strain energy function and the particular problem.

In the second paper (1980b), the complementary energy principle and the principle of stationary potential energy were applied to obtain lower and upper bounds on the total strain energy in two problems. For the one-dimensional problem of the all-around extension of a plane sheet with a circular hole, close bounds on the total strain energy were obtained for two forms of the strain energy function, and accurate estimates for the stress resultant at the edge of the sheet were obtained. This problem was treated previously by Rivlin and Thomas (1950) by a numerical approach. The problem of the large extension and torsion of a long elastic cylinder which is bonded at the ends to rigid plates was discussed next, and an approach was described for estimating the resultant end loads from estimates of the total strain energy derived from the variational principles. As an illustration, accurate estimates for the twisting moment and the axial force were obtained for elliptical cylinders with axes in the ratios of 2:1 and 4:1 for a wide range of extension and twist, the neo-Hookean form of the strain energy being assumed for the material of the cylinders.

It may be noted that in Lee and Shield (1980b), the leading term on the right-hand side of Eq. (2.10) should be μ^2.

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