such comments are as follows. First, Humphreys has presented
dynamic results for shallow arches showing symmetric behavior in
a region of the geometry parameters where asymmetric static
behavior would be expected to govern. These results suggest that
asymmetric dynamic behavior is possible above static geometric
limits. Second, comparisons between conical and spherical caps
has presented results in the paper for a rigidly clamped cone are applicable for values of $\lambda$ beyond the
static clamped spherical limit of 5.5. Thus, the author will
stand on the higher estimated limits suggested in the paper;
clearly, however, complete dynamic solutions including asym-
metry are required for precise establishment of these values.

11 J. S. Humphreys, "Dynamic Deformation States of Curved
Beams Under Impulsive Loads," JOURNAL OF APPLIED MECHANICS,

The Vibration of Unsymmetrical Rotating Shafts

L. G. Jaeger.1 The author has presented a paper which is ad-
mirably clear, concise, and thought-provoking. The writer would like to have his comments on the following approach to the free-
motion problem:

Beginning with equations (22) and (23), for the complementary
function, seek solutions of the form $\eta = Qe^{kx}$ and $\xi = k\eta$, where $k$
is a constant (possibly complex). Then equations (22) and (23)
of the paper may be written

$$k\lambda^3 + (\alpha + \beta)\lambda + (\omega_r^2 - \Omega^2)k - 2\omega_0 - k\mu = 0$$

$$\lambda^3 + (\alpha + \beta)\lambda + (\omega_r^2 - \Omega^2) + 2\omega_0 - k\mu = 0$$

Multiplying the first of these by $k$ and adding, and then multiply-
ing the second by $k$ and subtracting, gives

$$(1 + k^2)(\lambda^2 + (\alpha + \beta)\lambda - \Omega^2) + (\omega_r^2 + \omega_0^2)k = 0$$

$$(\omega_r^2 - \omega_0^2)k + (1 + k^2)(2\lambda + \alpha)\Omega = 0$$

Undamped Motion $\alpha = \beta = 0$

By (22b), $\lambda$ is real positive if $k$ is real negative. Let $k = -\tan \theta$, where $\tan \theta$ is real positive, and investigate the class of unstable motions which occur. Then

$$1 + k^2 = \sec^2 \theta$$

and

$$k \frac{1}{1 + k^2} = -\sin \theta \cos \theta = -\frac{\sin 2\theta}{2}$$

Equations (22a) and (22b) then become

$$\lambda^2 - \Omega^2 + \omega_0^2 = 0$$

$$\lambda = (\omega_r^2 - \omega_0^2) \frac{\sin 2\theta}{4\Omega}$$

where

$$\omega_r^2 - \omega_0^2 \sin^2 \theta + \omega_{0r}^2 \cos^2 \theta$$

The physical significance of the angle $\theta$ is thus apparent.
Since $\omega_0^2$ is proportional to $I_1$ and $\omega_r^2$ to $I_2$, $\omega_r^2$ is proportional to

(I sin^2 $\theta + I_2$ cos^2 $\theta$), which is the second moment of area for bending about an axis making an angle $\theta$ with the major principal axis. The first of equations (22c) involves this moment of inertia and the second of them involves the product of inertia with respect to this axis and its orthogonal partner. The unstable motion consists of bending about this axis increasing exponentially with time, and this motion is rectilinear relative to the rotating axes $\Omega$, $\omega_r$, Fig. 1 of this Discussion; the displacements $\xi$ and $\eta$ are components of this displacement. Eliminating $\lambda$ between equations (22c) and solving the resulting biquadratic for its real solution, one finds

$$\Omega^2 = 1 \left\{ \omega_r^2 + \left[ (\omega_r^2)^2 + (\omega_{0r}^2 - \omega_0^2) \frac{\sin^2 2\theta}{4} \right]^\frac{1}{2} \right\}$$

The solutions for $\Omega^2$ may be displayed conveniently on a Mohr's circle type of diagram shown in Fig. 2, in which the abscissa is an axis of (angular velocity)$^2$. With the notation there used, $\Omega^2 = \frac{1}{2}(OA + [OA^2 + AB^2])^{1/2}$

i.e.,

$$\Omega^2 = \frac{1}{2}(OA + OB)$$

Thence, using the first of equations (22c), $\lambda^2 = \frac{1}{2}(OB - OA)$. Letting $\lambda$ range over values 0 to $\pi/2$, all of which correspond to $\lambda$ positive, instability is encountered over the whole range of angular velocities between $\omega_r^2$ and $\omega_{0r}^2$. Drawing any vertical line intersecting the Mohr's circle, one may identify the axis about which the incremental bending takes place and the angular velocity at which it occurs. It may be noted that, for only slightly unsymmetrical shafts, $\Omega$ is very closely $\omega_r$.

If the succession of Mohr's circles corresponding to $r = 1, 2, 3,$
The positions for which this is stable motion. Using the first of equations (22d), we have

\[ \left( \lambda + \frac{\alpha}{2} \right)^2 + \left( \omega^2 - \alpha^2 \right) = 0 \]

Thence, eliminating \([\lambda + (\alpha/2)]\) and solving,

\[ \Omega^2 = \frac{1}{2} \left\{ \left( \omega^2 - \frac{\alpha^2}{4} \right) + \left( \omega^2 - \frac{\alpha^2}{4} \right)^2 \right\} \]

Again using a Mohr’s circle diagram and using an auxiliary point \(O'\) where \(OO' = \alpha^2/4\) (Fig. 4), we find

\[ \Omega = \frac{1}{2}(O'A + O'B) \]

wherein \(O'B\) is positive and \(O'A\) is accounted positive or negative according to whether \(O'\) is to the left or right of \(A\), respectively.

In this case, not all points \(B\) on the circle correspond to unstable motion. Using the first of equations (22d), we have

\[ \left( \lambda + \frac{\alpha}{2} \right)^2 = \frac{1}{2}(O'A + O'B) - (OA - OO') = \frac{1}{2}(O'B - O'A) \]

Thence, for instability, we must have \(\frac{1}{2}(O'B - O'A) = \alpha^2/4 = 0\); i.e., \(O'B \geq O'A + 200'\). The positions \(B\) for which this is satisfied define the reduced-width instability bands referred to by the author.

The form of equations (22d) suggests that the corresponding characteristic equation for \(\lambda\), equation (31), should reduce to a biquadratic in \((\lambda + \alpha/2)\). This is readily checked, and in fact we find

\[ \left( \lambda + \frac{\alpha}{2} \right)^2 + \left( \omega^2 - \omega'^2 + 2\Omega - \frac{\alpha^2}{4} \right) \left( \lambda + \frac{\alpha}{2} \right) + \left( \omega^2 - \Omega^2 - \frac{\alpha^2}{4} \right) \left( \omega'^2 - \Omega^2 - \frac{\alpha^2}{4} \right) = 0 \]

Insertion of the condition that \((\lambda + \alpha/2)\) shall be real and \(\geq \alpha/2\) leads to the condition

\[ \Omega^4 - (\omega^2 + \omega'^2 - \alpha^2)\Omega^2 + \omega^2\omega'^2 \leq 0 \]

which corresponds to the stability inequality (32) in the paper.

There appears to be a misprint in equation (35) of the paper, in which the writer believes the term \(4\omega^2\omega'^2\) should take a minus sign; viz.,

\[ \Omega^2 - (\omega^2 + \omega'^2 - \alpha^2)\Omega + \omega^2\omega'^2 \leq 0 \]

Since the criteria developed in the foregoing on the basis of \(\lambda\) real positive agree exactly with the author’s stability criteria, the writer had expected to reach identical conclusions on the position and width of the unstable bands but this is not the case. This is the only serious difference on which he wishes to comment.

The writer disagrees with the requirement \(\alpha \leq \omega^2 + \omega'\), given in expression (30) for the following reasons:

Write

\[ f(\Omega) = \Omega^4 - (\omega^2 + \omega'^2 - \alpha^2)\Omega^2 + \omega^2\omega'^2 \]

and suppose that for some value \(\alpha = \alpha_0\), \(f(\Omega)\) is positive for \(0 \leq \Omega \leq 0\). Then for another value \(\beta_0\) greater than \(\alpha_0\), \(f(\Omega)\) is a fortiori positive in the same range. Hence it follows that an increase in damping cannot introduce instabilities which were absent in the case of the smaller damping. The writer thinks there is a mistake in the paper here which arises from the omission of the condition that for an instability we must have \(\Omega^2 > 0\); i.e., a real shaft speed. It will be noticed that the characteristic equations for \(\lambda\) involve \(\Omega^2\), and thus the inclusion of the Routh criteria without including the additional condition \(\Omega^2 > 0\) admits the possibility of the whole range of values of \(\Omega\) from \(-\infty\) to \(+\infty\).

Thus it is believed that, although \(f(\Omega)\) as defined in the foregoing can be made negative by taking sufficiently large \(\alpha\), the region in which \(f(\Omega)\) is negative is entirely made up of negative values of \(\Omega^2\). For example, if \(\alpha > \omega^2 + \omega'\), then inspection of equation (35a) shows \(\Omega^2\) and \(\Omega'\) both to be negative.

The correct interpretation is thought to be that, as \(\alpha\) increases from zero, the unstable bands diminish in width and eventually vanish in the first mode, then in the second mode, and so on.
Internal Damping Present

If $\beta \neq 0$, the governing equation (39) does not in general reduce to a biquadratic. Instabilities can exist with $\lambda$ having a positive real part and nonzero imaginary part. An approach on the lines given in the foregoing is still possible, with $k$ now being complex, but it is probably simpler in this case to stay with the characteristic equation for $\lambda$ and use the Routh criteria, as the author does in the paper. It is suspected that the nature of such unstable motions may be as follows:

If $k$ is complex, then it is possible to find a pair of orthogonal axes at some angle $\phi$ to the principal axes of bending such that the ratio of displacement along these axes, say $\xi/\eta$, is purely imaginary. The movements $\xi$ and $\eta$ would then be in quadrature, and the displaced centroid of the cross section would be situated on an ellipse rotating with the $O\alpha, O\beta$-axes, having $\xi_0$ and $\eta_0$ as principal axes, the semiaxes of the ellipse increasing exponentially with time. If this is so, then the rectilinear motions referred to in the first two portions of this Discussion would be particular cases in which the ellipse has one of its principal axes zero, thus collapsing into a straight line.

A final minor point. In the case $\alpha = 0, \beta = 0$, a nondecaying solution $\lambda = \Omega$ exists for the characteristic equation (29) when

$$\Omega^2 = \frac{\omega_0^2 \eta_0 \gamma^2}{2(\alpha^2 + \omega_2^2)}$$

Thus any forcing term at this frequency produces an unbounded partial integral, giving the secondary critical speeds referred to in the paper. Where the existing term is due to gravity (as it always is in practice), it suffices that there shall be a nonzero gravity component at right angles to the shaft axis so that any nonvertical shaft may exhibit this kind of instability.

Summary

If $\beta = 0$, the characteristic equations for $\lambda$ are biquadratic in $(\lambda + \alpha/2)$. A class of rectilinear instability modes exists corresponding to $\lambda$ having a positive real part and no imaginary part. This class appears to cover all of the instabilities in free motion when internal damping is absent. A Mohr's circle type of diagram can then be used to identify the direction of the unstable displacement at any shaft speed $\Omega$ and the value of $\lambda$ associated with it.

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Traveling Force on a Timoshenko Beam

SING-CHIH TANG. At the end of this paper, the author mentioned that the discontinuity of velocity at $x = x_r$ except for the case $V = c_r$, was constant as the load position changed. He did not say what the constant was and how the discontinuity of velocity at $x = x_r$ increased indefinitely as the load traveled along the beam with $V = c_r$.

It is very simple to find the discontinuity at $x = x_r$ by use of the method of characteristics to solve this hyperbolic partial differential equation. One may also find the discontinuity by expansion of the Laplace inversion integral (18) immediately after the shear wave front. Problem 1 in the paper for $V = c_r$ is taken as an illustration.

From equation (18) of the paper, one has

$$w(x, \tau) = \frac{1}{2\pi i} \int e^{1/2} \left[ A e^{-\lambda \xi} + A e^{-\lambda \xi} + C e^{-\rho \xi} \right] p e^{\rho \xi} dp$$

Since the middle term in the foregoing equation is due to the contribution of the bending wave front, which has greater speed than that of the shear wave, the integral due to this term is continuous at $x = x_r$. The discontinuity of $w_r$ is only due to the rests. Since $c$ is an arbitrary constant greater than the real part of all singularities of the integrand, $c$ may be taken arbitrarily large, say $N$. Therefore

$$w_r = \frac{1}{2\pi i} \int e^{1/2} \left[ \begin{array}{c} A e^{-\lambda \xi} \\
A e^{-\lambda \xi} \\
C e^{-\rho \xi} \end{array} \right] p e^{\rho \xi} dp$$

Because one wishes to find the jump of $w_r$ at $x = x_r$ only, the integrand is truncated after the term of order $1/p$. Therefore

$$(A e^{-\lambda \xi} + C e^{-\rho \xi} e^{\rho \xi})$$

One expands $A_1, A_1$, and $e^{-\lambda \xi}$ in the power of $1/p$ as follows:

$$A_1 = \frac{k}{2} \left[ -\frac{1 + \alpha \gamma}{p} + \gamma(\gamma - 1) - \frac{1}{p^2} \right]$$

$$A_1 = \frac{k}{2} \left[ -\frac{1 + \gamma(\gamma - 1)}{p} + \gamma(\gamma - 1) - \frac{1}{p^2} \right]$$

$$e^{-\lambda \xi} = \exp \left\{ -\gamma^{1/2} \frac{\xi}{p} \right\}$$

Therefore

$$\Delta w_r = \frac{1}{2\pi i \sqrt{\pi}} \left[ \frac{e^{1/2}}{2} \left( \frac{\xi^2}{8(\gamma - 1)} - \frac{1}{p} \right) \right] e^{\rho(\tau - \gamma^{1/2} \xi)}$$

Finally, the jump of $w_r$ at $x = x_r$ is

$$\Delta w_r = \frac{1}{2\pi i \sqrt{\pi}} \left[ \frac{e^{1/2}}{2} \left( \frac{\xi^2}{8(\gamma - 1)} - \frac{1}{p} \right) \right] e^{\rho(\tau - \gamma^{1/2} \xi)}$$

where $\delta$ is the Dirac $\delta$-function. By the same procedure, $\Delta w_s$, in